Erhaltungsgrößen

Die Lagrangefunktion habe eine kontinuierliche Symmetrie:

$$L(q(t,s), \dot q(t,s), t+s\tau) = L(q(t,0), \dot q(t,0), t)$$

für alle s

Noether-Theorem:

$$I \equiv \frac{\partial L}{\partial \dot q}\frac{dq}{ds}\Big|_{s=0} - \left(\frac{\partial L}{\partial \dot q}\dot q - L\right)\tau \quad \text{ist zeitunabhängig.}$$

Beweis:

$$\frac{dI}{dt} = \underbrace{\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)}_{=\frac{\partial L}{\partial q}}\frac{dq}{ds} + \frac{\partial L}{\partial \dot q}\frac{d}{dt}\frac{dq}{ds} - \Bigg[\cancel{\underbrace{\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)}_{=\frac{\partial L}{\partial q}\ \text{(Lagrange-Gl.)}}\dot q} + \cancel{\frac{\partial L}{\partial \dot q}\ddot q} - \cancel{\frac{\partial L}{\partial q}\dot q} - \cancel{\frac{\partial L}{\partial \dot q}\ddot q} - \frac{\partial L}{\partial t}\Bigg]\tau$$

$$= \frac{\partial L}{\partial q}\frac{dq}{ds} + \frac{\partial L}{\partial \dot q}\frac{d\dot q}{ds} + \frac{\partial L}{\partial t}\frac{dt}{ds} - \cancel{\frac{\partial L}{\partial t}\underbrace{\frac{dt}{ds}}_{=\tau}} + \cancel{\frac{\partial L}{\partial t}\tau}$$

$$= \frac{dL}{ds} = 0 \quad \text{wegen Symmetrie} \qquad \blacksquare$$

Zeittranslationsinvarianz: $t \to t + s\tau$

$$q(t,s) = q(t)$$

$$L(q, \dot{q}, t + s\tau) = L(q, \dot{q}, t) \quad \text{für alle } s$$

$$I = -\Big(\underbrace{\frac{\partial L}{\partial \dot{q}}}_{= p} \dot{q} - L\Big)\underset{\substack{\uparrow \\ \tau = 1}}{\tau} = -(p\dot{q} - L) = -H$$

$\Longrightarrow H = E_{tot}$ ist zeitunabhängig

<u>Energieerhaltung</u>.

Räumliche Translationsinvarianz: $t \to t$

$$q(t,s) = (\underline{r}_1(t,s), \ldots, \underline{r}_N(t,s)) \qquad \underline{r}_i(t,s) = \underline{r}_i(t) + s\,\underline{e}$$

$\underline{e}$ = beliebiger Einheitsvektor

Paarpotenzial

$$V(q(t,s)) = \sum_{i,j=1}^{N} f_{ij}(|\underline{r}_i - \underline{r}_j|) = V(q(t)) \quad \text{für alle } s$$

$$L = T - V = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\underline{r}}_i^2 - V$$

$$I = \frac{\partial L}{\partial \dot{q}} \frac{dq}{ds} = \Big(\sum_{i=1}^{N} \underbrace{m_i \dot{\underline{r}}_i}_{= p_i}\Big) \cdot \underline{e} = p_{tot} \cdot \underline{e}$$

$\underline{e}$ beliebig $\Longrightarrow p_{tot} = \text{konst.}$ <u>Impulserhaltung</u>

Rotationsinvarianz: $q(t,s) = \underbrace{R(\underline{e},s)}\,\underline{r}(t)$

Rotation um Achse $\underline{e}$
mit Drehwinkel s

Infinitesimale Drehung $R(\underline{e},ds)\,\underline{r} = (\underline{e}\wedge\underline{r})\,ds$

$$\Rightarrow \left.\frac{dR(\underline{e},s)\,\underline{r}}{ds}\right|_{s=0} = \left.\frac{R(\underline{e},ds)}{ds}\right|_{s=0} = \underline{e}\wedge\underline{r}$$

$$I = \left.\frac{\partial L}{\partial \dot q}\,\frac{dq}{ds}\right|_{s=0} = \sum_{i=1}^{N} \underline{p}_i \cdot (\underline{e}\wedge\underline{r}_i) = \underline{e}\cdot\underbrace{\sum_{i=1}^{N}(\underline{r}_i\wedge\underline{p}_i)}$$

$\equiv$ Drehimpuls $= \underline{L}$

$\underline{e}$ beliebig $\Rightarrow \underline{L} = \sum_{i=1}^{N} \underline{r}_i\wedge\underline{p}_i = \text{konst.}$

Drehimpulserhaltung